You can evaluate limits three ways

1. Numerically
2. Graphically
3. Algebraically
It appears that the values get closer to 4, whether you approach 2 from the left (1.9, 1.99, 1.999) or from the right (2.1, 2.01, 2.001).

The answer is 4
Estimating a Limit Numerically

You do the table in your calculator

Use a table to estimate numerically the limit: \[ \lim_{{x \to 0}} \frac{x}{\sqrt{x + 1} - 1} \]

Approaching 0 from the left

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>1.9487</td>
</tr>
<tr>
<td>-0.01</td>
<td>1.995</td>
</tr>
<tr>
<td>-0.001</td>
<td>1.9995</td>
</tr>
<tr>
<td>1E-4</td>
<td>1.9999</td>
</tr>
</tbody>
</table>

Approaching 0 from the right

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>2.0488</td>
</tr>
<tr>
<td>.01</td>
<td>2.005</td>
</tr>
<tr>
<td>.001</td>
<td>2.0005</td>
</tr>
<tr>
<td>1E-4</td>
<td>2.00004999725</td>
</tr>
</tbody>
</table>
You do ....

Use a table to estimate numerically the limit: \( \lim_{x \to 3} (5x - 3) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9</td>
<td>11.500</td>
</tr>
<tr>
<td>2.99</td>
<td>11.950</td>
</tr>
<tr>
<td>2.999</td>
<td>11.995</td>
</tr>
<tr>
<td>3.0</td>
<td>?</td>
</tr>
<tr>
<td>3.001</td>
<td>12.005</td>
</tr>
<tr>
<td>3.01</td>
<td>12.05</td>
</tr>
<tr>
<td>3.1</td>
<td>12.5</td>
</tr>
</tbody>
</table>

The limit appears to be 12.

Do this one in your calculator.

Estimate the limit:

\[
\lim_{x \to 2} \frac{x^3 - 2x^2 + 2x - 4}{x - 2}
\]

The limit appears to be 12.

The limit appears to be 6.
Using a Graphing Utility to Estimate a Limit

Estimate the limit: \( \lim_{{x \to 1}} \frac{x^3 - x^2 + x - 1}{x - 1} \).

**Numerical Solution**

<table>
<thead>
<tr>
<th>X</th>
<th>Y_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.9998</td>
</tr>
<tr>
<td>1</td>
<td>1.0001</td>
</tr>
<tr>
<td>1.01</td>
<td>2.002</td>
</tr>
<tr>
<td>1.02</td>
<td>2.004</td>
</tr>
<tr>
<td>1.03</td>
<td>2.0059</td>
</tr>
</tbody>
</table>

**Graphical Solution**

Both numerically and graphically, it appears the limit is 2.

Text example: should have taken values closer to 1
Using a Graph to Find a Limit

Find the limit of \( f(x) \) as \( x \) approaches 3, where \( f \) is defined as

\[
f(x) = \begin{cases} 
2, & x \neq 3 \\
0, & x = 3
\end{cases}
\]

\[
\lim_{{x \to 3}} f(x) = 2.
\]

You do....

Find the limit of \( f(x) \) as \( x \) approaches 5, where \( f \) is defined as

\[
f(x) = \begin{cases} 
1, & x \neq 5 \\
-2, & x = 5
\end{cases}
\]

The limit of \( f(x) \) as \( x \) approaches 5 is 1, as shown on the graph.
Limits That Fail to Exist

Comparing Left and Right Behavior

The limit DOES NOT EXIST because the left-hand limit does not equal the right-hand limit.
The limit DOES NOT EXIST because the y-values grow unboundedly as we approach 0 from the left and right. We would write that the limit is INFINITY which means it does not exist because of unbounded growth.
Show that the limit does not exist.

$$\lim_{x \to 0} \frac{x}{|x|}$$

As shown on the graph,

$$f(x) = \frac{x}{|x|} = 1$$ for \( x > 0 \) and

$$f(x) = \frac{x}{|x|} = -1$$ for \( x < 0 \). This

implies that the limit does not exist.

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^4}$$

As \( x \) approaches 0 from either the left or the right, \( f(x) = \frac{1}{x^4} \) increases without bound, as shown. Because \( f(x) \) is not approaching a unique real number \( L \) as \( x \) approaches 0, you can conclude that the limit does not exist.
Since the y-values jump around seemingly randomly and do not approach a y-value as you get closer to 0, this limit DOES NOT EXIST.
Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist if any of the following conditions is true.

1. $f(x)$ approaches a different number from the right side of $c$ than it approaches from the left side of $c$.  
   Example 6
2. $f(x)$ increases or decreases without bound as $x$ approaches $c$.  
   Example 7
3. $f(x)$ oscillates between two fixed values as $x$ approaches $c$.  
   Example 8

Problems for you…..

Pages 788-789

#1-33 odds
Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$. In such cases, it is said that the limit can be evaluated by **direct substitution**. That is,

$$\lim_{x \to c} f(x) = f(c). \quad \text{Substitute } c \text{ for } x.$$  

Such functions are “well-behaved” for that limit. The limit equals the function value.

The trickier limits occur where there is some kind of discontinuity (break) in the curve or where the definition of the function changes at the $x$-value of interest (for instance, piece-wise functions.)
Basic Limits

Let $a$ and $c$ be real numbers and let $n$ be a positive integer.

1. $\lim_{x \to a} b = b$
2. $\lim_{x \to a} x = c$
3. $\lim_{x \to a} x^n = c^n$  [See the proof on page 835.]
4. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{c}$, for $n$ even and $c > 0$

Trigonometric functions can also be included in this list. For instance,

$\lim_{x \to \pi} \sin x = \sin \pi$

$= 0$

and

$\lim_{x \to 0} \cos x = \cos 0$

$= 1$

Properties of Limits

Let $a$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

$\lim_{x \to a} f(x) = L$  and  $\lim_{x \to a} g(x) = K$

1. Scalar multiple: $\lim_{x \to a} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to a} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to a} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to a} [f(x)]^n = L^n$
Direct Substitution and Properties of Limits

You can use the properties of limits and direct substitution to evaluate each limit.

a. \( \lim_{x \to 4} x^2 = (4)^2 = 16 \)
   
   **Direct Substitution**

b. \( \lim_{x \to 4} 5x = 5 \lim_{x \to 4} x = 5(4) = 20 \)
   
   **Scalar Multiple Property**

c. \( \lim_{x \to \pi} \frac{\tan x}{x} = \lim_{x \to \pi} \frac{\tan x}{\lim_{x \to \pi} x} = \frac{0}{\pi} = 0 \)
   
   **Quotient Property**

d. \( \lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3 \)

e. \( \lim_{x \to \pi} (x \cos x) = \left( \lim_{x \to \pi} x \right) \left( \lim_{x \to \pi} \cos x \right) = \pi(\cos \pi) = -\pi \)
   
   **Product Property**

f. \( \lim_{x \to 3} (x + 4)^2 = \left[ \left( \lim_{x \to 3} x \right) + \left( \lim_{x \to 3} 4 \right) \right]^2 = (3 + 4)^2 = 7^2 = 49 \)
   
   **Sum and Power Properties**
Find each limit.

a. \( \lim_{x \to 12} x^3 \)  = 8

b. \( \lim_{x \to 4} 8x \)  = 32

c. \( \lim_{x \to 16} \sqrt[4]{x} \)  = 2

d. \( \lim_{x \to \frac{\pi}{4}} \tan x \)  = -1
**Exploration**

Sketch the graph of each function. Then find the limits of each function as \( x \) approaches 1 and as \( x \) approaches 2. What conclusions can you make?

a. \( f(x) = x + 1 \)

b. \( g(x) = \frac{x^2 - 1}{x - 1} \)

c. \( h(x) = \frac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2} \)

Use a graphing utility to graph each function above. Does the graphing utility distinguish among the three graphs? Write a short explanation of your findings.

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{(1)^2 - 1}{1 - 1} = \frac{0}{0}, \quad \text{undefined}
\]

\[
\lim_{x \to 2} \frac{x^2 - 1}{x - 1} = \frac{(2)^2 - 1}{2 - 1} = \frac{3}{1} = 3
\]

The limit when \( x \to 1 \) is undefined, but the limit everywhere else exists.

No, the graph of all three functions appears to be one line, the line \( f(x) = x + 1 \).

The limit when \( x \to 1 \) and \( x \to 2 \) is undefined.
The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

**Limits of Polynomial and Rational Functions**

1. If \( p \) is a polynomial function and \( c \) is a real number, then
   \[
   \lim_{{x \to c}} p(x) = p(c). \quad \text{(See the proof on page 835.)}
   \]

2. If \( r \) is a rational function given by \( r(x) = \frac{p(x)}{q(x)} \), and \( c \) is a real number such that \( q(c) \neq 0 \), then
   \[
   \lim_{{x \to c}} r(x) = r(c) = \frac{p(c)}{q(c)}.
   \]

**Evaluating Limits by Direct Substitution**

Find each limit.

\[
\lim_{{x \to -1}} (x^2 + x - 6)
\]

\[
\lim_{{x \to -1}} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = \frac{6}{2} = 3
\]

\[
\lim_{{x \to -1}} (x^2 + x - 6) = (-1)^2 + (-1) - 6
\]

\[
= -6
\]
Find each limit.

\[
\lim_{x \to 2} (x^2 + 5x + 4)
\]

\[
\lim_{x \to 2} \frac{x^2 + 5x + 4}{x + 4}
\]

**Exploration**

Use a graphing utility to graph the function

\[
f(x) = \frac{x^2 - 3x - 10}{x - 5}.
\]

Use the trace feature to approximate \(\lim_{x \to 5} f(x)\). What do you think \(\lim_{x \to 5} f(x)\) equals? Is \(f\) defined at \(x = 5\)? Does this affect the existence of the limit as \(x\) approaches 5?

Exploration

\[
\lim_{x \to 4} f(x) = 6, \lim_{x \to 5} f(x) = 7
\]

No; no
Problems for you ..... 
Pages 788-790 
#35 - 69 odds
Algebraic techniques that is.

**Dividing Out Technique**

This is the first thing to do if you get 0/0 using *direct substitution*

\[
\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}
\]

Note that direct substitution gives you 0/0, time to rework the quotient.

\[
\begin{align*}
\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \\
&= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \\
&= \lim_{x \to -3} (x - 2) \\
&= -5
\end{align*}
\]

Now you can use direct substitution on the simplified form.
We can check our result both numerically and graphically.

Numerically

Graphically
Another example.....

\[
\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}
\]

Note, again, that direct substitution results in 0/0.
Rework the limit.

Now, use direct substitution.

\[
\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)}
\]

Factor denominator.

\[
= \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)}
\]

Divide out common factor.

\[
= \lim_{x \to 1} \frac{1}{x^2 + 1}
\]

Simplify.

\[
= \lim_{x \to 1} \frac{1}{x^2 + 1}
\]

Direct substitution.

\[
= \frac{1}{1^2 + 1}
\]

Simplify.

\[
= \frac{1}{2}
\]

Simplify.

\[
f(x) = \frac{x - 1}{x^3 - x^2 + x - 1}
\]

is undefined when \( x = 1 \).

\[
(f(1), \frac{1}{2})
\]
Find the limit: \( \lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2} \).

Find the limit: \( \lim_{x \to 2} \frac{x - 2}{x^3 - 2x^2 + 2x - 4} \).
Rationalizing Technique

We “rationalize” limits that give 0/0 on direct substitution and have a radical in them.

Note: direct substitution yields 0/0.

\[
\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{0}{0}
\]

0/0 is an indeterminate form, as is infinity/infinity.

\[
\frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{x+1} - 1}{x} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}
\]

\[
= \frac{(x + 1) - 1}{x(\sqrt{x+1} + 1)}
\]

\[
= \frac{x}{x(\sqrt{x+1} + 1)}
\]

\[
= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0
\]

In this form, direct substitution works.
We can check this algebraic solution by checking the limit numerically and graphically.

### Numeric Check

<table>
<thead>
<tr>
<th>$x$</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.5132</td>
<td>0.5013</td>
<td>0.5001</td>
<td>?</td>
<td>0.4999</td>
<td>0.4988</td>
<td>0.4881</td>
</tr>
</tbody>
</table>

### Graphic Check

$f(x) = \frac{\sqrt{x + 1} - 1}{x}$ is undefined when $x = 0$. The graph shows the behavior near $x = 0$. The limit as $x$ approaches 0 is $\frac{1}{2}$.
You do a rationalization….

Extra Example 2

Find the limit: \( \lim_{x \to 0} \frac{\sqrt{x+9} - 3}{x} \).

Check your answer numerically and graphically.
The dividing out and rationalizing techniques may not work well for finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of these types of functions.

Approximate the limit: \( \lim_{x \to 0} (1 + x)^{1/x} \).

Doesn’t fit the “factor (divide)” out or the “rationalization” techniques

Numerical estimate

Graphical estimate
You do .......

Approximate the limit:
\[ \lim_{x \to 1} \frac{x^2 - 4x - 5}{x + 1}. \]

The table indicates that the limit as \( x \) approaches \(-1\) is \(-6\).

Approximate the limit:
\[ \lim_{x \to 0} \frac{1 - \cos x}{x}. \]

The graph indicates that the limit as \( x \) approaches \(0\) is \(0\).
Problems for you.....
Page 798

#1-27 odds
We have covered these extensively already. Remember to recognize the superscript.

Finding One-Sided Limits

Find the limit of \( f(x) \) as \( x \) approaches 1.

\[
f(x) = \begin{cases} 
4 - x, & x < 1 \\
4x - x^2, & x > 1 
\end{cases}
\]

Because the one-sided limits both exist and are equal to 3, it follows that

\[
\lim_{x \to 1^-} f(x) = 3.
\]
Comparing Limits from the Left and Right

\[ f(x) = \begin{cases} 
17.80, & 0 < x \leq 1 \\
19.20, & 1 < x \leq 2 \\
20.60, & 2 < x \leq 3 
\end{cases} \]

\[ \lim_{x \to 2^-} f(x) = 19.20 \]

\[ \lim_{x \to 2^+} f(x) = 20.60. \]

Because these one-sided limits are not equal, the limit of \( f(x) \) as \( x \to 2 \) does not exist.
A Limit from Calculus

Evaluating a Limit from Calculus

For the function given by \( f(x) = x^2 - 1 \), find

\[
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}
\]

Direct substitution produces an indeterminate form.

\[
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - [(3)^2 - 1]}{h}
\]

\[
= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h}
\]

\[
= \lim_{h \to 0} \frac{6h + h^2}{h}
\]

\[
= 6 + 0
\]

By factoring and dividing out, you obtain the following.

\[
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} \frac{h(6 + h)}{h}
\]

\[
= \lim_{h \to 0} (6 + h)
\]

\[
= 6 + 0
\]

\[
= 6
\]

So, the limit is 6.
For the function given by \( f(x) = 2x^2 + 1 \), find
\[
\lim_{{h \to 0}} \frac{f(2 + h) - f(2)}{h}.
\]
Problems for you…

Pages 798-799

#29-59 odds
The red lines indicate the slopes at one point (tangent) on the curve NOT the slope between two points on the curve.
Visually Approximating the Slope of a Graph

Use the graph in Figure 11.24 to approximate the slope of the graph of \( f(x) = x^2 \) at the point \((1, 1)\).

\[
\text{Slope} = \frac{\text{change in } y}{\text{change in } x}
\]

\[
\approx \frac{2}{1} = 2.
\]

Because the tangent line at the point \((1, 1)\) has a slope of about 2, you can conclude that the graph of \(f\) has a slope of about 2 at the point \((1, 1)\).
Approximating the Slope of a Graph

Figure 11.25 graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the indicated point and give a physical interpretation of the result. (Source: National Climatic Data Center)

The graph of $f$ has a slope of about 3 at (1, 1).

We do....

\[
\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{-16}{2} = -8 \text{ degrees per month.}
\]

This means that you can expect the monthly normal temperature in November to be about 8 degrees lower than the normal temperature in October.
The graph below depicts the monthly normal temperatures in a local city. Approximate the slope of the graph at the indicated point, and give a physical interpretation of the result.

The graph indicates that the tangent line at the point has a slope of about $-10$ degrees per month. This means that you can expect the monthly normal temperature in November to be about 10 degrees lower than the normal temperature in October.
The limiting process......

A secant is a line that connects two points on a curve.

As \( h \) approaches 0, the secant line approaches the tangent line.
Using the limit process, you can find the exact slope of the tangent line at 
\((x, f(x))\).

**Definition of the Slope of a Graph**

The slope \(m\) of the graph of \(f\) at the point \((x, f(x))\) is equal to the slope of its
tangent line at \((x, f(x))\), and is given by

\[
m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided this limit exists.

**Finding the Slope of a Graph**

Find the slope of the graph of \(f(x) = x^2\) at the point \((-2, 4)\).

Find an expression that represents the slope of a secant line at \((-2, 4)\).

\[
m_{\text{sec}} = \frac{f(-2 + h) - f(-2)}{h}
= \frac{(-2 + h)^2 - (-2)^2}{h}
= \frac{4 - 4h + h^2 - 4}{h}
= \frac{-4h + h^2}{h}
= \frac{h(-4 + h)}{h}
= -4 + h, \ h \neq 0
\]

Set up difference quotient.

Substitute into \(f(x) = x^2\).

Expand terms.

Simplify.

Factor and divide out.

Simplify.

Next, take the limit of \(m_{\text{sec}}\) as \(h\) approaches 0.

\[
m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} (-4 + h) = -4 + 0 = -4
\]

The graph has a slope of \(-4\) at the point \((-2, 4)\).
Finding the Slope of a Graph

Find the slope of \( f(x) = -2x + 4 \).

\[
m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

Set up difference quotient.

\[
= \lim_{h \to 0} \frac{[-2(x + h) + 4] - (-2x + 4)}{h}
\]

Substitute into \( f(x) = -2x + 4 \).

\[
= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h}
\]

Expand terms.

\[
= \lim_{h \to 0} \frac{-2h}{h}
\]

Divide out.

\[
= -2
\]

Simplify.
Find the slope of the graph of $f(x) = x^3$ at the point $(2, 8)$.

Problems for you....
Page 808
#1-12 all
The derivative tells you the **slope** of a function, its *instantaneous rate of change*.

The derivative is also called “*f prime of x*”

The **Definition of the Derivative**

The derivative of \( f \) at \( x \) is given by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided this limit exists.
Finding a Derivative

Find the derivative of $f(x) = 3x^2 - 2x$.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{[3(x + h)^2 - 2(x + h)] - (3x^2 - 2x)}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}
\]

\[
= \lim_{h \to 0} \frac{h(6x + 3h - 2)}{h}
\]

\[
= \lim_{h \to 0} (6x + 3h - 2)
\]

\[
= 6x + 3(0) - 2
\]

\[
= 6x - 2
\]

So, the derivative of $f(x) = 3x^2 - 2x$ is $f'(x) = 6x - 2$. 
That first notation, Liebniz notation is used by mathematicians. Engineers use the y prime notation.

Find \( f'(x) \) for \( f(x) = \sqrt{x} \). Then find the slopes of the graph of \( f \) at the points \((1, 1)\) and \((4, 2)\) and equations of the tangent lines to the graph at the points.

\[
\begin{align*}
\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].
\end{align*}
\]

Rationalize!

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}
= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}
= \frac{1}{2\sqrt{x}}
\]

\[
f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x + h} - \sqrt{x}}{h}\right) \left(\frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}\right)
= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})}
= \frac{1}{2\sqrt{x}}
\]
Now, finding the slope at the given points….

At the point (1, 1), the slope is
\[ f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}. \]

At the point (4, 2), the slope is
\[ f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}. \]

Now find the tangent line

An equation of the tangent line at the point (4, 2) is
\[ y - y_1 = m(x - x_1) \]
\[ y - 2 = \frac{1}{4}(x - 4) \]
\[ y = \frac{1}{4}x + 1. \]
Find $f'(x)$ for $f(x) = \sqrt{x} + 1$. Then find the slopes of the graph of $f$ at the points $(4, 3)$ and $(9, 4)$.

$f'(x) = \frac{1}{2\sqrt{x}}$.

$f'(4) = \frac{1}{4}$.

$f'(9) = \frac{1}{6}$.
Problems for you.....
Pages 808-809
#13-37 odds (just part a)
# 39-45 odds
# 51, 53
Simply put, the limit at infinity equals the y-value of the horizontal asymptote.

If there is not a horizontal asymptote, then the limit at infinity is positive or negative infinity.
Definition of Limits at Infinity

If \( f \) is a function and \( L_1 \) and \( L_2 \) are real numbers, the statements
\[
\lim_{x \to -\infty} f(x) = L_1 \quad \text{Limit as } x \text{ approaches } -\infty
\]
and
\[
\lim_{x \to \infty} f(x) = L_2 \quad \text{Limit as } x \text{ approaches } \infty
\]
denote the limits at infinity. The first statement is read "the limit of \( f(x) \) as \( x \) approaches \( -\infty \) is \( L_1 \)," and the second is read "the limit of \( f(x) \) as \( x \) approaches \( \infty \) is \( L_2 \)."

To help evaluate limits at infinity, you can use the following definition.

Limits at Infinity

If \( r \) is a positive real number, then
\[
\lim_{x \to \infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the right}
\]
Furthermore, if \( x^r \) is defined when \( x < 0 \), then
\[
\lim_{x \to -\infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the left}
\]

Evaluating a Limit at Infinity

Find the limit.
\[
\lim_{x \to \infty} \left( 4 - \frac{3}{x^2} \right)
\]

Algebraic Solution

Use the properties of limits listed in Section 11.1.

\[
\lim_{x \to \infty} \left( 4 - \frac{3}{x^2} \right) = \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2}
\]

\[
= \lim_{x \to \infty} 4 - 3\left( \lim_{x \to \infty} \frac{1}{x^2} \right)
\]

\[
= 4 - 3(0)
\]

\[
= 4
\]

So, the limit of \( f(x) = 4 - \frac{3}{x^2} \) as \( x \) approaches \( \infty \) is 4.
Evaluating a Limit at Infinity

Find the limit.
\[ \lim_{x \to \infty} \left( 4 - \frac{3}{x^2} \right) \]

Graphical Solution

Comparing Limits at Infinity

\[ f(x) = \frac{-2x + 3}{3x^2 + 1} \]

\[ f(x) = \frac{-2x^2 + 3}{3x^2 + 1} \]

\[ f(x) = \frac{-2x^3 + 3}{3x^2 + 1} \]

\[ \lim_{x \to \infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x \to \infty} \frac{-2 + \frac{3}{x^2}}{3 + \frac{1}{x^2}} = -0 + 0 = 0 \]

\[ \lim_{x \to \infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x \to \infty} \frac{-2 + \frac{3}{x^2}}{3 + \frac{1}{x^2}} = -2 + 0 = -\frac{2}{3} \]

\[ \lim_{x \to \infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x \to \infty} \frac{-2x + \frac{3}{x^2}}{3 + \frac{1}{x^2}} = -2x + 0 = -\frac{2}{3} \]
General principles for limits at infinity for rational functions

• If the highest power is in the denominator, the function approaches 0 as x approaches infinity or negative infinity.

• If the highest power is in the numerator, the function grows without bound (some would say the limit is infinity). If degree is just one greater in numerator than denominator there is a SLANT ASYMPTOTE.

• If the numerator and denominator have equal high powers, then the function approaches the coefficients on the largest power.
Limits at Infinity for Rational Functions

Consider the rational function $f(x) = \frac{N(x)}{D(x)}$, where

$$N(x) = a_nx^n + \cdots + a_0 \quad \text{and} \quad D(x) = b_mx^m + \cdots + b_0.$$  

The limit of $f(x)$ as $x$ approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, the limit does not exist.

You do …

Find the limit as $x$ approaches each function.

a. $f(x) = \frac{-x + 4}{5x^2 + 2}$

\[ \lim_{x \to \infty} \frac{5x(x - 3)}{2x} \]

b. $f(x) = \frac{-x^2 + 4}{5x^2 + 2}$

\[ \lim_{x \to \infty} \frac{4x^3 - 5x}{8x^4 + 3x^2 - 2} \]

c. $f(x) = \frac{-x^3 + 4}{5x^2 + 2}$

\[ \lim_{x \to \infty} \frac{-6x^2 + 1}{3x^2 + x - 2} \]
Finding the Average Cost

You are manufacturing greeting cards that cost $0.50 per card to produce. Your initial investment is $5000, which implies that the total cost $C$ of producing $x$ cards is given by $C = 0.50x + 5000$. The average cost $\bar{C}$ per card is given by

$$\bar{C} = \frac{C}{x} = \frac{0.50x + 5000}{x}.$$

Find the average cost per card when (a) $x = 1000$, (b) $x = 10,000$, and (c) $x = 100,000$. (d) What is the limit of $\bar{C}$ as $x$ approaches infinity?

- When $x = 1000$, the average cost per card is
  $$\bar{C} = \frac{0.50(1000) + 5000}{1000} = \frac{5500}{1000} = $5.50.$$

- When $x = 10,000$, the average cost per card is
  $$\bar{C} = \frac{0.50(10,000) + 5000}{10,000} = \frac{55000}{10,000} = $1.00.$$

- When $x = 100,000$, the average cost per card is
  $$\bar{C} = \frac{0.50(100,000) + 5000}{100,000} = \frac{55000}{100,000} = $0.55.$$

As $x \to \infty$, the average cost per card approaches $0.50$.

As $x$ approaches infinity, the limit of $\bar{C}$ is

$$\lim_{x \to \infty} \frac{0.50x + 5000}{x} = 0.50.$$
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Do problems #3-45 (the multiples of 3)